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THE LINEARIZATION METHOD IN GEOMETRICAL INVERSE PROBLEMS OF THE THEORY OF ELASTICITY†

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Both rigorous and linearized formulations of the boundary-geometrical inverse problem (BIP) of the theory of elasticity are presented for an elastic half-space with a defect modelled by an elastic inclusion or cavity. When formulating the linearized BIP it is assumed that the boundary of the defect can be specified in a local system of coordinates connected with the boundary of the defect. The aim of the solution of the linearized problem is to find a scalar function, namely, the shape variation, i.e. the normal distance between the known boundary and the points of the boundary to be found. The plane and the anti-plane problems of the theory of elasticity are considered. Using the boundary integral equations the rigorous BIP can be reduced to a non-linear system of integro-differential equations, and the linearized BIP can be reduced to a linear system of integral equations. It is found that the linearized boundary integral equations preserve the order of singularities characteristic of the ordinary boundary integral equations. To solve the rigorous inverse problem an iteration procedure of successive approximations of the shape of the defect is proposed. In this case it suffices to solve a linearized BIP at each iteration step.

Problems involving the reconstruction of the shape of a defect in an elastic medium, based on information about the wave fields observed at the boundary of the medium belong to the class of boundary-geometrical [1] inverse problems (BIPs) of the theory of elasticity. Using boundary integral equations (BIEs), these problems can be reduced [2–7] to solving ill-posed problems [1–8] for complex non-linear systems of integro-differential equations.

The main success in solving these inverse problems is related to the use of algorithms for solving direct problems. As a rule, such methods involve an iterative procedure. A direct problem in which the shape of the defect is known is solved at each iteration step, followed by a comparison of the wave fields for the unknown and known defects. The comparison is made in a region on the boundary of the medium accessible to the observer, with a view to finding a new shape for which the wave field scattered by the defect and that measured at the boundary are close to one another (in uniform or mean square metrics).

The need to introduce a known defect, the boundary of which is to "contract" the desired boundary as a result of the iterative process, weakens the formulation of the BIP, but enables it to be solved by available mathematical methods.

1. FORMULATION OF THE DIRECT AND INVERSE PROBLEMS

Consider the following boundary-value problem for an elastic half-space V $(x_3 \leqslant 0)$ containing a cylindrical inclusion $V^{(1)} \subset V$, the generating line of the cylinder being parallel to the x_2 axis. We shall assume that both the inclusion $V^{(1)}$ and the external medium $V^{(0)} = V \backslash V^{(1)}$ are homogeneous and anisotropic, and the elastic-constant tensors $c^{(m)}$ and densities $\rho^{(m)}$, $m = 0, 1$ are known for each of them. The superscript $m = 1$ indicates the medium of the inclusion and $m = 0$ indicates the external medium. A vibrating source $p^*(x_1)e^{-i\omega t}$, $x_1 \in [a, b]$, acts on a section $\Lambda = \{(x_1, x_2, x_3): x_1 \in [a, b], x_2 \in (-\infty), x_3 = 0\}$ of the half-space, giving rise to vibrations in the half-space. The remaining part of the boundary $\partial V \backslash \Lambda$ will be assumed to be stress-free. The radiation conditions [9, 10] complete the formulation of the problem. The principle of limit absorption is used to state these conditions.

Assuming steady-state vibrations, on separating the time factor $e^{i\omega t}$, we can write the basic boundary-value problem as the equations of motion

$$\nabla \cdot \boldsymbol{\sigma}^{(m)} + \rho^{(m)}\omega^2 \mathbf{u}^{(m)} = 0, \quad x \in V^{(m)}, \quad m = 0, 1 \tag{1.1}$$

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the constitutive equations

$$\boldsymbol{\sigma}^{(m)} = \mathbf{c}^{(m)} \odot \nabla \mathbf{u}^{(m)}, \quad x \in V^{(m)}, \quad m = 0,\ 1 \tag{1.2}$$

the boundary conditions

$$\mathbf{e}_3 \cdot \boldsymbol{\sigma}^{(0)}|_{x_3=0} = \begin{cases} \mathbf{p}^*, & x \in \Lambda \\ 0, & x \in \partial V \setminus \Lambda \end{cases} \tag{1.3}$$

and the matching conditions

$$\mathbf{p}^{(0)} = \mathbf{p}^{(1)},\ \mathbf{u}^{(0)} = \mathbf{u}^{(1)},\ x \in S \tag{1.4}$$

at the boundary $S = \partial V^{(1)}$ of the inclusion ($\mathbf{p}^{(0)} = 0, x \in S$ in the case of a cavity).

Here $\boldsymbol{\sigma}^{(m)}$ and $\mathbf{u}^{(m)}$ are the stress tensor and displacement vector in $V^{(m)}$, $p^{(m)} = n \cdot \boldsymbol{\sigma}^{(m)}$ is the stress vector, $\mathbf{n}$ is the outward normal vector to the surface S, and $\mathbf{e}_i$ is the ith vector of the Cartesian basis.

In the formulation of the direct problem S is known, while the displacement field in the section $\Phi = \{(x_1, x_2, x_3) : x_1 \in [c, d], x_2 \in (-\infty, +\infty), x_3 = 0$ of the boundary of the half-space is to be determined. Conversely, in the formulation of the BIP the boundary S will need to be determined from information on the displacement field on Φ. Thus, the BIP requires an additional boundary condition

$$\mathbf{u}^{(0)} = \mathbf{f},\ x \in \Phi \tag{1.5}$$

where $\mathbf{f}$ is a given vector-valued function independent of x_2.

Under the condition described above the direct problem (1.1)–(1.4) and the BIP (1.1)–(1.5) for orthotropic materials split into two independent problems: the case of plane deformation and the anti-plane problem of the theory of elasticity. In (1.1)–(1.5) $\mathbf{u} = (u_1, 0, u_3)$, $\mathbf{p} = (p_1, 0, p_3)$, $\mathbf{p}^* = (\mathbf{p}_1^*, 0, p_3^*)$, $\mathbf{f} = (f_1, 0, f_3)$ in the plane case and $\mathbf{u} = (0, u_2, 0)$, $\mathbf{p} = (0, p_2, 0)$, $p^* = (0, p_2^*, 0)$, $\mathbf{f} = (0, f_2, 0)$ in the anti-plane case. The section of S and $V^{(m)}$ by the plane x_1x_3 will be denoted by l and $\Omega^{(m)}$, respectively. We shall confine ourselves to the case when l is a smooth simple contour.

2. REDUCTION OF THE DIRECT AND INVERSE PROBLEMS TO A SYSTEM OF BIEs

We introduce Green's tensors $\mathbf{U}_r^{(m)}(x, \xi)$ of Eqs (1.1) and (1.2) satisfying the equations

$$\nabla \cdot \boldsymbol{\Sigma}_r^{(m)} + \rho^{(m)}\omega^2 \mathbf{U}_r^{(m)} = -\mathbf{e}_r \delta(x,\ \xi),\ \ \boldsymbol{\Sigma}_r^{(m)} = \mathbf{c}^{(m)} \nabla \mathbf{U}_r^{(m)} \tag{2.1}$$

where $x, \xi \in R^2$ for $m = 1$ and $x, \xi \in R^2$ for $m = 0$, and for $m = 0$ also the boundary condition

$$\mathbf{e}_3 \cdot \boldsymbol{\Sigma}_r^{(0)}\Big|_{x_3=0} = 0 \tag{2.2}$$

as well as the radiation conditions. In (2.1) $\delta(x, \xi)$ is the two-dimensional delta-function.

Remark 1. In the plane case Green's tensors of Eqs (1.1) and (1.2) were constructed for orthotropic materials [5] as single Fourier integrals using boundary condition (2.2). In the anti-plane case the corresponding Green's functions for an anisotropic half-space can be expressed analytically [11] in terms of Hankel functions.

Using Green's functions and the reciprocity theorem [12], one can express the displacement field at any point in $\Omega^{(0)}$ and $\Omega^{(1)}$ in terms of the boundary values of the displacement field and the stress vector on l

$$u_r^{(0)}(\xi) = S_r^{(0)}[\mathbf{u}^{(0)},\ \mathbf{p}^{(0)},\ l,\ \xi] + u_r^*(\xi),\ \xi \in \Omega^{(0)} \tag{2.3}$$

$$u_r^{(1)}(\xi) = -S_r^{(1)}[\mathbf{u}^{(1)},\ \mathbf{p}^{(1)},\ l,\ \xi],\ \xi \in \Omega^{(1)} \tag{2.4}$$

Here we introduce the notation

$$S_r^{(m)}[\mathbf{u},\ \mathbf{p},\ l,\ \xi] = \int_l (\mathbf{P}_r^m(x,\ \xi)\cdot \mathbf{u}(x) - \mathbf{U}_r^{(m)}(x,\ \xi)\cdot \mathbf{p}(x))dl_x \tag{2.5}$$

for the Somigliana integral operator, where $\mathbf{P}^m{}_r = \mathbf{n}\cdot\boldsymbol{\Sigma}_r{}^{(m)}$ and $\mathbf{n} = (n_1, n_3)$ is the outward normal vector to l. In (2.3) $u^*{}_r$ is the displacement field in the half-space when there is no defect

$$u_r^*(\xi) = \int_a^b U_r^{(0)}(\zeta,\ \xi)\cdot \mathbf{p}^*(\zeta_1)d\zeta_1,\ \ \zeta = (\zeta_1,\ 0)$$

To obtain the BIE, the boundary l must be approached in the limit in integral representations (2.3) and (2.4) and boundary conditions (1.4) must be satisfied.

The limit value of the integral operator (2.5) can be represented as

$$\lim_{\xi\to y\in l} S_r^{(m)}[\mathbf{u},\mathbf{p},l,\xi] = S_r^{(m)}[\mathbf{u},\mathbf{p},l,y] + M[\mathbf{P}_r^{(m)}\cdot\mathbf{u} - \mathbf{U}_r^{(m)}\cdot\mathbf{p},y],\ \ \xi\in\Omega^{(m)} \tag{2.6}$$

The first term on the right-hand side of (2.6) is an integral in the sense of the Cauchy principal value [13], and the second term is the jump of a singular integral. The notation for the jump is chosen to emphasize its dependence on the integrand. To complete the jumps it suffices to use the following properties of Green's tensors.

Property 1. We represent Green's tensor $\mathbf{U}_r{}^{(0)}$ by two terms, the first being Green's tensor for an unbounded medium, while the second one is an additional term, which is strictly smooth in $x_3 < 0$ and has no effect on the limit properties of the Somigliana operator.

Property 2. The limit properties of the Somigliana operator are independent of $\rho\omega^2$, so that its jumps are identical with the results known in statics [13].

The following theorem is an extension of well-known theorems in potential theory [13].

Theorem 1. Let l be a closed Lyapunov contour and let $\mathbf{u}\in C^{(\gamma)}(l)$, $\mathbf{p}\in C(l)$, where $C^{(\gamma)}(l)$ is the space of complex-valued functions on l satisfying the Hölder condition with exponent γ. Then limit values of the Somigliana operator exist as $\xi\to y\in l$ ($\xi\in\Omega^{(m)}$, $m = 0, 1$) and can be represented in the form (2.6), the following formulae being satisfied for the jumps

$$M[\mathbf{U}_r^{(m)}\cdot\mathbf{p},y] = 0,\ \ M[\mathbf{P}_r^{(m)}\cdot\mathbf{u},y] = \frac{(-1)^m}{2}u_r(y) \tag{2.7}$$

Using this result, we arrive at the following system of BIEs

$$\frac{1}{2}u_r(y) = S_r^{(0)}[\mathbf{u},\ \mathbf{p},l,\ y] + u_r^*(y),\ \ \frac{1}{2} = -S_r^{(1)}[\mathbf{u},\ \mathbf{p},l,\ y],\ \ y\in l \tag{2.8}$$

where the functions $\mathbf{u}$ and $\mathbf{v}$ ($\mathbf{u} = \mathbf{u}^{(0)} = \mathbf{u}^{(1)}$, $\mathbf{p} = \mathbf{p}^{(0)} = \mathbf{p}^{(1)}, x\in l$) are introduced by virtue of boundary conditions (1.4).

System (2.8) is a resolving system for the direct problem. Computing the boundary fields $\mathbf{u}$ and $\mathbf{p}$ from the solution of this system, we can find the displacement field at any point of the half-space using (2.3) and (2.4). In particular, for points on the boundary of the half-space we have

$$u_r^{(0)}(\zeta) = S_r^{(m)}[\mathbf{u},\mathbf{p},l,\zeta] + u_r^*(\zeta),\ \ \zeta = (\zeta_1,\ 0) \tag{2.9}$$

We will now consider the BIP (1.1)–(1.5) for an unknown inclusion $\tilde{V}^{(1)}\subset V$. All the symbols referring to this problem will be indicated by a tilde. Combining system (2.8) and Eq. (2.9), written for the required contour $\tilde{l}$, and using boundary condition (1.5), we can reduce the BIP to the following non-linear system of integro-differential equations for $\tilde{\mathbf{u}}$ and $\tilde{\mathbf{p}}$, and an equation for the contour l

$$\begin{aligned} &f_r(\zeta) = S_r^{(0)}[\tilde{\mathbf{u}},\tilde{\mathbf{p}},\tilde{l},\zeta] + u_r^*(\zeta),\ \ \zeta\in\Phi \\ &\frac{1}{2}\tilde{u}_r(\tilde{y}) = S_r^{(0)}[\tilde{\mathbf{u}},\tilde{\mathbf{p}},\tilde{l},\tilde{y}] + u_r^*(\tilde{y}),\ \ \tilde{y}\in\tilde{l} \end{aligned} \tag{2.10}$$

$$\frac{1}{2}\tilde{u}_r(\tilde{y}) = -S_r^{(1)}[\tilde{\mathbf{u}}, \tilde{\mathbf{p}}, \tilde{l}, \tilde{y}], \quad \tilde{y} \in \tilde{l}$$

3. LINEARIZATION OF THE BIE

We shall assume that when formulating the BIP some a priori information is available about the shape and location of the unknown contour $\tilde{l}$, namely, that a contour l is given that $\tilde{l}$ can be expressed in a local system of coordinates connected with l using the relation

$$\tilde{x}_i = x_i + \nu(x)n_i(x), \quad i = 1, \ 3, \quad x \in l, \quad \tilde{x} \in \tilde{l} \tag{3.1}$$

We shall say that $\tilde{l}$ is close to l if the following conditions are satisfied

1. $\| k\nu \|_{C(l)} \ll 1$, where k is the largest wave number ($m = 0, 1$)
2. $\| \varkappa\nu \|_{C(l)} \ll 1$, where $\varkappa$ is the curvature of l
3. $\| \nu_{,s} \|_{C(l)} \ll 1$, $(\ )_{,s} = \mathbf{s} \cdot \nabla(\ )$, where $\mathbf{s}$ is a tangent vector to l.

A function ν satisfying conditions (1)–(3) will be called a variation of the shape of the contour l. Assuming that $\tilde{l}$ and l are close to one another, we linearize integral representations (2.3) and (2.4) in the neighbourhood of l

$$\tilde{u}_r^{(m)}(\xi) - u_r^{(m)}(\xi) = (-1)^m (S_r^{(m)}[\tilde{\mathbf{u}}, \tilde{\mathbf{p}}, \tilde{l}, \xi] - S_r^{(m)}[\mathbf{u}, \ \mathbf{p}, \ l, \ \xi]) \tag{3.2}$$

$$\xi \in \Omega^{(m)} \cap \tilde{\Omega}^{(m)}, \quad m = 0, \ 1$$

Henceforth, to fix our ideas, we shall assume that $\Omega^{(m)}$ and $\tilde{\Omega}^{(m)}$ are open domains and we shall omit the arguments $x \in l$, $\xi \in \Omega^{(m)} \cap \tilde{\Omega}^{(m)}$, for brevity. Using the identity

$$\tilde{n} d\tilde{l} = ((n - \nu s)_{,s}) dl$$

and an expansion of the form

$$\mathbf{\Sigma}_r^{(m)}(x + \nu\mathbf{n}) \cong \mathbf{\Sigma}_r^{(m)} + \mathbf{n} \cdot \nabla_{\tilde{x}} \mathbf{\Sigma}_r^{(m)}(x + \nu\mathbf{n})|_{\nu=0}\, \nu \cong \mathbf{\Sigma}_r^{(m)} + \mathbf{\Sigma}_{r,n}^{(m)} \nu$$

we linearize the singular part of the Somigliana operator (2.5)

$$\int_{\tilde{l}} \mathbf{P}_r^{(m)}(x + \nu\mathbf{n}) \cdot \tilde{\mathbf{u}}(x + \nu\mathbf{n}) d\tilde{l}_{\tilde{x}} \cong \int_l (\mathbf{n} - (\nu s)_{,\mathbf{s}}) \cdot (\mathbf{\Sigma}_r^{(m)} + \mathbf{\Sigma}_{r,\mathbf{n}}^{(m)} \nu) \cdot \tilde{\mathbf{u}}(x + \nu\mathbf{n}) dl_x \cong$$

$$\cong \int_l (\mathbf{n} \cdot \mathbf{\Sigma}_r^{(m)} + \nu\mathbf{n} \cdot \nabla \mathbf{\Sigma}_r^{(m)} \cdot \mathbf{n} - (\nu s \cdot \mathbf{\Sigma}_r^{(m)})_{,\mathbf{s}} + \nu s \cdot \nabla \mathbf{\Sigma}_r^{(m)} \cdot \mathbf{s}) \cdot \tilde{\mathbf{u}}(x + \nu\mathbf{n}) dl_x =$$

$$= \int_l \mathbf{P}_r^{(m)} \cdot \tilde{\mathbf{u}}(x + \nu\mathbf{n}) dl_x + \int_l ((\nabla \cdot \mathbf{\Sigma}_r^{(m)}) \cdot \tilde{\mathbf{u}}(x + \nu\mathbf{n}) + \mathbf{s} \cdot \mathbf{\Sigma}_r^{(m)} \cdot \tilde{\mathbf{u}}_{,s}(x + \nu\mathbf{n})) \nu dl_x \tag{3.3}$$

We introduce the variation of the boundary displacement field

$$\delta\mathbf{u}(x) = \tilde{\mathbf{u}}(x + \nu\mathbf{n}) - \mathbf{u}(x), \quad x \in l$$

Using (2.1), we rewrite the linearized representation (3.3) in the form

$$\int_{\tilde{l}} \mathbf{P}_r^{(m)}(x + \nu\mathbf{n}) \cdot \tilde{\mathbf{u}}(x + \nu\mathbf{n}) d\tilde{l}_{\tilde{x}} \cong \int_l \mathbf{P}_r^{(m)} \cdot (\mathbf{u} + \delta\mathbf{u}) dl_x +$$

$$+ \int_l (-\rho^{(m)} \omega^2 \mathbf{U}_r^{(m)} \cdot (\mathbf{u} + \delta\mathbf{u}) + \mathbf{H}_r^{(m)} \cdot (\mathbf{u} + \delta\mathbf{u})_{,\mathbf{s}}) \nu dl_x$$

$$\mathbf{H}_r^{(m)} = \mathbf{s} \cdot \mathbf{\Sigma}_r^{(m)} \tag{3.4}$$

Now we linearize the regular part of the Somigliana operator (2.5)

$$\int_{\tilde{l}} \mathbf{U}_r^{(m)}(x + \nu\mathbf{n}) \cdot \tilde{\mathbf{p}}(x + \nu\mathbf{n}) d\tilde{l}_x \cong \int_l (\mathbf{U}_r^{(m)} + \mathbf{U}_{r,n}^{(m)} \nu) \cdot \tilde{\mathbf{p}}(x + \nu\mathbf{n}) J dl_x =$$

$$= \int_l \mathbf{U}_r^{(m)} \cdot (\mathbf{p} + \delta\mathbf{p}) dl_x + \int_l \mathbf{U}_{r,\mathbf{n}}^{(m)} \cdot (\mathbf{p} + \delta\mathbf{p}) \nu dl_x \tag{3.5}$$

where J is the Jacobian of the transformation of the length element of the arc of the contour $\tilde{l}$ into the length element of the arc of the contour l, and where the variation

$$\delta\mathbf{p}(x) = \tilde{\mathbf{p}}(x+\nu\mathbf{n})J(x) - \mathbf{p}(x), \quad x \in l$$

of the boundary stress field is introduced. Substituting (3.4) and (3.5) into (3.2), we have

$$\begin{aligned}
&\tilde{u}_r^{(m)}(\xi) - u_r^{(m)}(\xi) \cong (-1)^m (S_r^{(m)}\\
&[\delta\mathbf{u}, \delta\mathbf{p}, l, \xi] + W_r^{(m)}[\mathbf{u}+\delta\mathbf{u}, \mathbf{p}+\delta\mathbf{p}, \nu, l, \xi])\\
&\xi \in \Omega^{(m)} \cap \tilde{\Omega}^{(m)}, \ m = 0, 1
\end{aligned} \tag{3.6}$$

Here

$$W_r^{(m)}[\mathbf{u}, \mathbf{p}, \nu, l, \xi]) = \int_l G_r^{(m)}(x, \xi)\nu(x)dl_x \tag{3.7}$$

$$G_r^{(m)} = -\rho^{(m)}\omega^2 \mathbf{U}_r^{(m)}(x,\xi)\cdot\mathbf{u}(x) + \mathbf{H}_r^{(m)}(x,\xi)\mathbf{u}_{,s}(x) - \mathbf{U}_{r,\mathbf{n}}^{(m)}(x,\xi)\cdot\mathbf{p}(x)$$

Remark 2. As can be seen from (3.6) and (3.7), the linearization procedure does not give rise to any hypersingular linear operators as $\xi \to y \in l$.

We continue analytically the left- and right-hand sides of (3.6) to the region $\Omega^{(0)} \cap \tilde{\Omega}^{(1)}$ ($m = 0$) and $\Omega^{(1)} \cap \tilde{\Omega}^{(0)}$ ($m = 1$). Then their limit values on l are related by the approximate equality

$$\begin{aligned}
&\delta u_r(y) - (u_r^{(m)}(y) + \delta u_r^{(m)}(y))_{,\mathbf{n}}\nu(y) \cong\\
&\cong (-1)^m \lim_{\xi\to y}(S_r^{(m)}[\delta\mathbf{u}, \delta\mathbf{p}, l, \xi] + W_r^{(m)}[\mathbf{u}+\delta\mathbf{u}, \mathbf{p}+\delta\mathbf{p}, \nu, l, \xi])\\
&y \in l, \ \xi \in \Omega^{(m)}
\end{aligned} \tag{3.8}$$

By Remark 2 the integrals in (3.8) will be understood as the principal values in Cauchy's sense [13]. We shall compute the jumps of the integral operators on the right-hand side of (3.8). For the first term, using (2.7), we have

$$M[\mathbf{P}_r^{(m)}\cdot\delta\mathbf{u} - \mathbf{U}_r^{(m)}\cdot\delta\mathbf{p}, y] = \frac{(-1)^m}{2}\delta u_r(y) \tag{3.9}$$

The following theorem provides a formula for computing the jump of the operator $W_r^{(m)}$.

Theorem 2. Let l be a Lyapunov contour, let $\mathbf{u}^{(m)}$ be complex-valued functions smooth in a neighbourhood of l ($\xi \in \Omega^{(m)}$, $m = 0, 1$) that satisfy the matching conditions (1.4) on l, and let ν be a smooth scalar function on l. Then limit values of the operator $W_r^{(m)}$ in (3.7) exist and the following formula holds for the jump

$$M[\mathbf{G}_r^{(m)}\nu, y] = \frac{(-1)^m}{2}u_{r,\mathbf{n}}^{(m)}(y)\nu(y) \tag{3.10}$$

Proof. We consider the following scalar function on l

$$\begin{aligned}
&C(\mathbf{U}_r^{(m)}(x,\xi), \mathbf{u}^{(m)}(x)) = \mathbf{H}_r^{(m)}(x,\xi)\cdot\mathbf{u}_{,s}^{(m)}(x) + \mathbf{P}_r^{(m)}(x,\xi)\cdot\mathbf{u}_{,n}^{(m)}(x)\\
&x \in l, \ \ \xi \in \Omega^{(m)}
\end{aligned}$$

It can be shown that

$$C(\mathbf{U}_r^{(m)}(x,\xi), \mathbf{u}^{(m)}(x)) = C(\mathbf{u}^{(m)}(x), \mathbf{U}_r^{(m)}(x,\xi))$$

Then

$$M[G_r^{(m)}\nu, y] = M[(-\rho^{(m)}\omega^2 \mathbf{U}_r^{(m)} \cdot \mathbf{u} + \mathbf{H}_r^{(m)} \cdot \mathbf{u}_{,s} - \mathbf{U}_{r,}^{(m)} \cdot \mathbf{p})\nu, y] =$$
$$= M[(-\rho^{(m)}\omega^2 \mathbf{U}_r^{(m)} \cdot \mathbf{u} + \mathbf{U}_{r,s}^{(m)} \cdot \mathbf{h}^{(m)} - \mathbf{P}_r^{(m)} \cdot \mathbf{u}_{,n}^{(m)})\nu, y] =$$
$$= -M[\mathbf{U}_r^{(m)} \cdot (\rho^{(m)}\omega^2 \mathbf{u}\nu + (\mathbf{h}^{(m)}\nu)_{,s}), y] - M[\mathbf{P}_r^{(m)} \cdot \mathbf{u}_{,n}^{(m)}\nu, y]$$
$$\mathbf{h}^{(m)} = \mathbf{s} \cdot \boldsymbol{\sigma}^{(m)}$$

Applying (2.7), we obtain (3.10).

Using (3.9) and (3.10), we rewrite the approximate equality (3.8) as

$$\frac{1}{2}(\delta u_r(y) - (u_r^{(m)}(y) + \delta u_r^{(m)}(y))_{,\mathbf{n}}\nu(y)) \cong$$
$$\cong (-1)^m (S_r^{(m)}[\delta\mathbf{u}, \delta\mathbf{p}, l, y] + W_r^{(m)}[\mathbf{u} + \delta\mathbf{u}, \mathbf{p} + \delta\mathbf{p}, \nu, l, y]), \quad y \in l \tag{3.11}$$

Remark 3. It has not been required that the boundary fields should be small to obtain (3.6) or (3.11). As $\nu \to 0$ relation (3.11) changes to a system of BIEs for $\delta\mathbf{u}$ and $\delta\mathbf{p}$ similar to system (2.8) for $\mathbf{u}$ and $\mathbf{p}$ when there is no external load. It follows that $\delta\mathbf{u}$ and $\delta\mathbf{p}$ tend to the null vector as $\nu \to 0$.

For $m = 0$, we let ξ in (3.6) tend to the point $\zeta \in \Phi$ on the boundary of the half-space, and we use boundary condition (1.5). Replacing the approximate equality by the rigorous one in the resulting expression and in (3.11), we combine them to form a joint system. To within higher-order infinitesimals, we have

$$\begin{gathered} f_r(\zeta) - u_r^{(0)}(\zeta) = S_r^{(0)}[\delta\mathbf{u}, \delta\mathbf{p}, l, \zeta] + W_r^{(0)}[\mathbf{u}, \mathbf{p}, \nu, l, \zeta] \\ \frac{1}{2}(\delta u_r(y) - u_{r,\mathbf{n}}^{(0)}(y)\nu(y)) = S_r^{(0)}[\delta\mathbf{u}, \delta\mathbf{p}, l, y] + W_r^{(0)}[\mathbf{u}, \mathbf{p}, \nu, l, y] \\ \frac{1}{2}(\delta u_r(y) - u_{r,\mathbf{n}}^{(0)}(y)\nu(y)) = -S_r^{(1)}[\delta\mathbf{u}, \delta\mathbf{p}, l, y] - W_r^{(1)}[\mathbf{u}, \mathbf{p}, \nu, l, y] \\ \xi \in \Phi, \quad y \in l \end{gathered} \tag{3.12}$$

We will state this result as a theorem.

Theorem 3. The variation of the shape of l and the variations of the boundary fields on l and on the boundary of the half-space are related by the system of special BIEs (3.12).

Remark 4. In the case of a cavity-type defect it suffices to put $p \equiv \delta p \equiv 0$ in the systems of BIEs (2.8) and (3.12). It is not necessary to use the last equation in either of these systems.

4. THE USE OF LINEARIZED BIEs TO SOLVE INVERSE GEOMETRIC PROBLEMS

The inverse problem of finding the variation of the shape ν will be called the linearized BIP. To solve it, it is necessary to solve the direct problem for a known contour l and compute the boundary fields $\mathbf{u}$, $\mathbf{u}_{,s}$, $\mathbf{p}$, $\mathbf{u}_{,\mathbf{n}}^{(0)}$, $\mathbf{u}_{,\mathbf{n}}^{(1)}$ on l and $\mathbf{u}^{(0)}$, $\xi \in \Phi$ on the boundary of the half-space. This enables one to construct the integral kernels in (3.12), which is a linear system of integral equations for $\delta\mathbf{u}$, $\delta\mathbf{p}$ and ν. Since the upper equations in (3.12) are integral equations with smooth kernels, the solution of (3.12) is an ill-posed problem, while the remaining equations in (3.12) are singular.

If the condition that ν should be small is violated, then to solve the non-linear problem we propose to set up an iterative procedure of successive approximations. The linearized BIP for some initial approximation $l_0 = l$ is solved first and a function ν_0 is computed, so that one can proceed to the next approximation l_1, etc.

Remark 5. In the case of the analytic problem for an anisotropic half-space with a defect, the system of BIEs (3.12) becomes the system of BIEs obtained in [7], where a numerical analysis of the efficiency of the proposed method of solving the BIP was given.

Remark 6. System (3.12) is of interest in the direct formulation when investigating the effect of small variations of the shape of a defect on the variation of the wave field in a half-space.

Remark 7. The proposed linearization method can be extended to three-dimensional problems for anisotropic semi-bounded and bounded media. The main obstacle in using this approach is the problem of constructing Green's functions which are appropriate from the point of view of a numerical realization.

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